Grade 9/10 Math Circles
March 20, 2024
Probability I - Solutions

In-Lesson Exercises

1. No, because AB is not an element of {A,B,C}.
2. We take every combination of elements: $\emptyset$, $\{x\}$, $\{y\}$, $\{z\}$, $\{x, y\}$, $\{y, z\}$, $\{x, z\}$, S
3. Many possible answers
4. Since every element in A is also in B, $A \cap B = A$ and $A \cup B = B$.
5. (a) $E \cap P = \{2, 4, 6\} \cap \{2, 3, 5\} = \{2\}$
   (b) $O \cap L = \{1, 3, 5\} \cap \{1, 2\} = \{1\}$
   (c) $E \cup L = \{2, 4, 6\} \cup \{1, 2\} = \{1, 2, 3, 5\}$
   (d) $P \cup O = \{2, 3, 5\} \cup \{1, 3, 5\} = \{1, 2, 3, 5\}$

   Bonus: $O \cap E = \{1, 3, 5\} \cap \{2, 4, 6\} = \emptyset$, so O and E are disjoint.
6. Since each roll is equally likely, $P(E) = n(E)/6$.
   (a) $P(A) = n(A)/6 = n(\{2\})/6 = 1/6$
   (b) $P(B) = n(B)/6 = n(\{1, 3, 5\})/6 = 3/6$
   (c) $P(A \cap B) = n(A \cap B)/6 = n(\{2\} \cap \{1, 3, 5\})/6 = n(\emptyset)/6 = 0$
   (d) $P(A \cup B) = n(A \cup B)/6 = n(\{2\} \cup \{1, 3, 5\})/6 = n(\{1, 2, 3, 5\})/6 = 4/6$
7. The probabilities must sum to 1, so P(blue) = 0.5.

Additional Exercises

1. There are $6 \cdot 6 = 36$ possible outcomes from rolling two dice. There are 6 ways to roll doubles, so $P(\text{doubles}) = 6/36$.

   To determine the probability that a roll is less than 11, we will use the complement rule. There are only three ways to roll at least 11: 5/6, 6/5, and 6/6. If L is the event that a roll is at least 11, then

$$P(L) = 1 - P\left(L^C\right) = 1 - \frac{n\left(L^C\right)}{36} = 1 - \frac{3}{36} = \frac{33}{36}$$

2. We can use the union rule for this problem.

   Let A be the event that you draw an ace and S be the event that you draw a spade. There are 52 cards, 4 aces, and 13 spades. Exactly 1 card is both an ace and a spade.

$$P(A \cup S) = P(A) + P(S) - P(A \cap S) = \frac{n(A)}{52} + \frac{n(S)}{52} - \frac{n(A \cap S)}{52} = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52}$$

   Now, let F be the event that you draw a face card. There are 12 face cards, 3 of which are spades. So,

$$P(F \cup S) = P(F) + P(S) - P(F \cap S) = \frac{n(F)}{52} + \frac{n(S)}{52} - \frac{n(F \cap S)}{52} = \frac{12}{52} + \frac{13}{52} - \frac{3}{52} = \frac{22}{52}$$

3. We start with a total of $3 + 6 + 1 = 10$ chocolates. Let D represent dark chocolate and S represent the sample space of all chocolates.

   The probability of drawing a dark chocolate will be the number of dark chocolates divided by the total number of chocolates. However, adding k dark chocolate increased both the amount of dark chocolate and the amount of total chocolate. That is,

$$P(D) = \frac{n(D)}{n(S)} = \frac{1+k}{10+k}$$

   We must set the above equation equal to $1/4$ and solve for k.

$$\frac{1+k}{10+k} = \frac{1}{4}$$

Cross multiplying and solving gives us $k = 2$.

4. Let R, B, O, T be the events that the ball is red, blue, has a 1, and has a 2, respectively.

   We want to find $P(R \cap O)$, $P(R \cap T)$, $P(B \cap O)$, and $P(B \cap T)$.

   Rearranging the union rule lets us solve for the first value:

$$P(R \cap O) = P(R) + P(O) - P(R \cup O) = \frac{10}{20} + \frac{9}{20} - \frac{13}{20} = \frac{6}{20}$$

   This tells us that there are 6 balls which are both red and have a 1. Since every ball has a number, the remaining 4 red balls must have a 2. (Notice that we are using the fact that $O = T^C$.) Thus, $P(R \cap T) = \frac{4}{20}$.

   We know there are 9 total balls with a 1, and 6 of those are red. The remaining 3 must be blue, so $P(B \cap O) = \frac{3}{20}$.

   Similarly, there are 11 total balls with a 2, and 4 of those are red. The remaining 7 must be blue, so $P(B \cap T) = \frac{7}{20}$.

   To check our work, we can confirm that the probability of every possibility adds up to 1:

$$P(R \cap O) + P(R \cap T) + P(B \cap O) + P(B \cap T) = \frac{6}{20} + \frac{4}{20} + \frac{3}{20} + \frac{7}{20} = 1$$